

自然数 N のベル表現の個数について?

journal or publication title	東京商船大学研究報告. 自然科学
volume	53
page range	1-6
year	2002
URL	http://id.nii.ac.jp/1342/00000564/

On the Number of Pellian Representations of Natural Number N II

OSAMU MATSUSHITA

Abstract

Pell sequence is defined by the second order linear recurrence such that $P_{n+1} = 2P_n + P_{n-1}$ with $P_0 = 1, P_1 = 1$ and its terms are called Pell numbers. It is known that any positive integer N can be represented by the sum of Pell numbers as follows: $N = \sum_{i=1}^n \alpha_i P_i$ where $0 \leq \alpha_i \leq 2$. Such a representation is called a Pellian representation of N .

We denote the number of Pellian representations of N by $R(N)$. The purpose of this paper are to show some properties of $R(N)$ and the method of computing $R(N)$ by using those properties. We set $\sum_{k=1}^n P_k = S_n$. Main results are as follows;

- (1) $R(S_n) = 1$
- (2) $R(P_n) = \left\lceil \frac{n+2}{2} \right\rceil$
- (3) $R(S_n + 1) = R(S_n - 1) = 2$
- (4) $0 \leq M \leq \left\lceil \frac{P_{n+1}}{2} \right\rceil \implies R(S_n + M) = R(S_{n+1} - M)$
- (5) $S_n < S_n + M < P_{n+1} \implies R(S_n + M) = R(S_n + M - P_n)$
- (6) $P_{n+1} < S_n + M < P_{n+1} + P_n \implies R(S_n + M) = R(S_n + M - P_{n+1}) + R(S_n + M - 2P_n)$
- (7) $P_{n+1} < S_n + M \leq S_n + \left\lceil \frac{P_{n+1}}{2} \right\rceil \implies R(S_n + M) = R(S_n + M - P_{n+1}) + R(S_n + M - 2P_n)$.

1 Pell numbers and Zeckendorf's theorem

Fibonacci sequence $\{F_n\}$ is defined by the second order linear recurrence such that $F_{n+1} = F_n + F_{n-1}$ with $F_0 = 0$ and $F_1 = 1$, and the number appearing in the sequence is called a Fibonacci number. A great amount of theorems have been discovered by many mathematicians since 13th century when Leonard Pisa, Fibonacci, described the Fibonacci number in his famous book 'Liber abati'. But Zeckendorf's theorem which we shall state is comparatively new.

Theorem 1.1 (Zeckendorf⁽²⁾⁽³⁾). *Any positive integer are represented uniquely as follows:*

$$N = \sum_{i=1}^n \alpha_i F_i \quad (\alpha_i = 0 \text{ or } 1, \alpha_i \alpha_{i+1} = 0, \alpha_1 = 0)$$

This theorem asserts that any natural number N can be uniquely represented by using Fibonacci numbers with the condition such that $\alpha_i = 0$ or 1 , $\alpha_i \alpha_{i+1} = 0$, $\alpha_1 = 0$. If we don't assume that $\alpha_i \alpha_{i+1} = 0$, the representation is not unique in general. Such a representations of N is called a Fibonacci representation of N . We shall denote the numbers of the Fibonacci representations of N by $R_F(N)$. There have been a lot of investigations^{(4),(5),(6),(7),(8)} of the property $R_F(N)$.

Definition 1.2. *Pell sequence $\{P_n\}$ is the sequence which is satisfied the linear recurrence relation such that $P_{n+1} = 2P_n + P_{n-1}$ with the condition $P_0 = 0$, $P_1 = 1$. Each P_n is a called Pell number.*

Zeckendorf theorem for Pell numbers is as follows.

Theorem 1.3. ^{(9),(10)} *Every positive integer has a unique representation in the form*

$$N = \sum_{i=1}^n \alpha_i P_i$$

where $\alpha_i = 0, 1$ or 2 , $\alpha_i = 2 \Rightarrow \alpha_{i-1} = 0$.

Similarly to Fibonacci numbers if we do not suppose the condition that $\alpha_i = 2 \Rightarrow \alpha_{i-1} = 0$, the representation is not unique in general. Such a representation is called a Pellian representation.

Henceforce we shall denote the number of representations of N by $R(N)$.

2 The property of $R(N)$

First of all we shall sum up the results in the author's previous paper⁽¹⁾.

Definition 2.1. $S_n = P_1 + \cdots + P_n$.

Lemma 2.2. *For $n \geq 1$, we have $S_n = \frac{P_{n+1} + P_n - 1}{2}$.*

Theorem 2.3. $N = S_n \iff R(N) = 1$.

Theorem 2.4. $0 \leq M \leq \left\lfloor \frac{P_{n+1}}{2} \right\rfloor \implies R(S_n + M) = R(S_{n+1} - M)$.

Theorem 2.5. *For $n = 0, 1, 2, 3, \dots$, $R(P_n) = \left\lfloor \frac{n+2}{2} \right\rfloor$ ($n = 0, 1, 2, \dots$) where $R(0) = 1$.*

The proof of these theorems can be seen in Matsushita⁽¹⁾.

Lemma 2.6. $S_n + S_{n-1} + 1 = P_{n+1}$

We can easily prove this lemma by using lemma 2.2.

Theorem 2.7. $S_n < S_n + M < P_{n+1} \implies R(S_n + M) = R(S_n + M - P_n)$

Proof. The Pellian representation of $S_n + M$ contains P_n . This claim can be shown as follows.

Since $P_n + M < P_{n+1}$, the Pellian representation of $S_n + M$ does not contain P_{n+1} .

Moreover from lemma 2.2,

$$\begin{aligned} 2(P_1 + P_2 + \cdots + P_{n-1}) &= P_n + P_{n-1} - 1 \\ &< S_n \\ &< S_n + M. \end{aligned}$$

Hence the Pellian representation of $S_n + M$ must contains P_n .

Therefore from the equation $S_n + M = P_n + (S_n + M - P_n)$, we completes the proof. \square

Remark. Since $S_n - P_n = S_{n-1}$ and $P_{n+1} - S_n - 1 = S_{n-1}$, the above theorem can be written as follows;

$$0 \leq M \leq S_{n-1} \implies R(S_n + M) = R(S_{n-1} + M)$$

From this remark we can prove the next theorem easily.

Theorem 2.8. $R(S_n + 1) = R(S_n - 1) = 2$.

Proof. By using the formula in the above remark repeatedly, we obtain $R(S_n + 1) = R(S_1 + 1) = R(2) = 2$. On the other hand from theorem 2.4, we have $R(S_n - 1) = R(S_{n-1} + 1)$.

Hence, $R(S_n - 1) = 2$. \square

When $P_{n+1} < S_n + M < P_n + P_{n+1}$, we have the following theorem.

Theorem 2.9. $P_{n+1} < S_n + M < P_{n+1} + P_n \implies R(S_n + M) = R(S_n + M - P_{n+1}) + R(S_n + M - 2P_n)$.

To prove the above theorem we need the following lemmas .

Lemma 2.10. When $P_{n+1} < N$, if P_{n+1} is not used in a Pellian representation of N , then P_n is used twice in the representation.

Proof. Since $P_{n+1} < N$ and from lemma 2.2,

$$\begin{aligned} N - P_n > P_{n+1} - P_n &= P_n + P_{n-1} \\ &> 2(P_1 + P_2 + \cdots + P_{n-1}) \end{aligned}$$

Hence P_n must be used in the representation of $N - P_n$. □

Lemma 2.11. Let $P_{n+1} < S_n + M < P_{n+1} + P_n$.

\implies

(1) If we use P_{n+1} for representing $S_n + M$, we don't use P_n .

(2) If we do not use P_{n+1} for representing $S_n + M$, we use P_n twice.

Proof. Since $S_n + M < P_{n+1} + P_n$, we have $S_n + M - P_{n+1} < P_n$. So (1) is obvious. (2) is already proved in lemma 2.10. □

From lemma 2.10 and lemma 2.2, we can easily prove the theorem 2.9.

Lemma 2.12. $S_n + \left\lfloor \frac{P_{n+1}}{2} \right\rfloor < P_{n+1} + P_n$

Proof. From lemma 2.2,

$$\begin{aligned} S_n + \left\lfloor \frac{P_{n+1}}{2} \right\rfloor &\leq \frac{P_n + P_{n+1} - 1}{2} + \frac{P_{n+1}}{2} \\ &= P_{n+1} + \frac{P_n - 1}{2} \\ &< P_{n+1} + P_n. \end{aligned}$$

□

Corollary 2.13. $P_{n+1} < S_n + M \leq S_n + \left\lfloor \frac{P_{n+1}}{2} \right\rfloor$

$$\implies R(S_n + M) = R(S_n + M - P_{n+1}) + R(S_n + M - 2P_n)$$

3 Computation of $R(N)$

The results we obtained in the previous section are as follows;

(1) $R(S_n) = 1$

(2) $R(P_n) = \left\lfloor \frac{n+2}{2} \right\rfloor$

(3) $R(S_n + 1) = R(S_n - 1) = 2$

(4) $0 \leq M \leq \left\lfloor \frac{P_{n+1}}{2} \right\rfloor \implies R(S_n + M) = R(S_{n+1} - M)$

$$(5) S_n < S_n + M < P_{n+1} \implies R(S_n + M) = R(S_n + M - P_n)$$

$$(6) P_{n+1} < S_n + M < P_{n+1} + P_n \implies R(S_n + M) = R(S_n + M - P_{n+1}) + R(S_n + M - 2P_n)$$

$$(7) P_{n+1} < S_n + M \leq S_n + \left\lceil \frac{P_{n+1}}{2} \right\rceil \implies R(S_n + M) = R(S_n + M - P_{n+1}) + R(S_n + M - 2P_n).$$

Combining these results we can compute $R(N)$.

Below is the table of P_n and S_n for $1 \leq n \leq 10$. Now using this table we shall compute $R(N)$ for some natural numbers N .

n	1	2	3	4	5	6	7	8	9	10
P_n	1	2	5	12	29	70	169	408	985	2378
S_n	1	3	8	20	49	119	288	696	1681	4059

Example 3.1. $R(1000)$

$$P_9 = 985 < 1000 = 696 + 304 = S_8 + 304. \text{ So, we have from (5)}$$

$$\begin{aligned} R(1000) &= R(1000 - 985) + R(1000 - 2 \cdot 408) \\ &= R(15) + R(184). \end{aligned}$$

Now we compute $R(15)$, $R(184)$.

$$R(15) = R(20 - 5) = R(8 + 5) = R(13 - 12) + R(13 - 10) = R(1) + R(3) = 1 + 1 = 2$$

$$R(184) = R(119 + 65) = R(184 - 169) + R(184 - 140) = R(15) + R(44) = 2 + R(44).$$

$$\text{And } R(44) = R(49 - 5) = R(20 + 5) = R(20 - 5) = R(15) = 2.$$

Finally we obtain $R(1000) = 6$.

Example 3.2. $R(2000)$.

$$P_9 = 985 < 2000 = 1681 + 319 = S_9 + 319. \text{ So we get from (5)}$$

$$\begin{aligned} R(2000) &= R(1681 + 319) \\ &= R(696 + 319) \\ &= R(1015 - 985) + R(1015 - 2 \cdot 408) \\ &= R(30) + R(199). \end{aligned}$$

Now we compute

$$R(199) = R(119 + 80) = R(199 - 169) + R(199 - 140) = R(30) + R(59)$$

and

$$R(59) = R(49 + 10) = R(20 + 10) = R(30).$$

Therefore, $R(2000) = 3 R(30)$.

$$R(30) = R(20 + 10) = R(30 - 29) + R(30 - 24) = R(1) + R(6).$$

$$\text{And } R(6) = R(8 - 3) = R(5) = R(P_3) = \left\lfloor \frac{3+1}{2} \right\rfloor = 2.$$

Hence $R(30) = 1 + 2 = 3$ and finally we have $R(2000) = 3 \cdot 3 = 9$.

Example 3.3. $R(800)$

$$\begin{aligned} R(800) &= R(696 + 104) = R(288 + 104) = R(119 + 104) \\ &= R(288 - 65) = R(119 + 65) = R(184 - 169) + R(184 - 140) \\ &= R(15) + R(44) = R(15) + R(15) = 4 \end{aligned}$$

References

- (1) Matsushita, O.: On the number of Pellian representations of natural number N , Journal of the Tokyo of Mercantile Marine, pp11-19, No.52(2001)
- (2) Zeckendorf, E.: Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas, Bull.Soc.Roy.Sci.Liège, Vol.41,179-82,(1972)
- (3) Hogatt, V.E.Jr.: Fibonacci and Lucas Numbers, Houghton Mifflin Company,(1969),pp 69-78
- (4) Klarner, D. A.: Representation of N as a sum of distinct elements from special sequences, The Fibonacci Quarterly, Vol.4.4,pp.289-305,(1966)
- (5) Klarner, D. A.: Partitions of N into distinct Fibonacci numbers, The Fibonacci Quarterly, Vol.6.4,pp.235-44,(1968)
- (6) Carlitz, L.: Fibonacci representations, The Fibonacci Quarterly, Vol.6.4,pp.193-220,(1968)
- (7) Bicknell, M.et al.: The Number of representations of N using distinct Fibonacci numbers, counted by recursive formulas, The Fibonacci Quarterly, Vol.37.1,pp47-60,(1999)
- (8) Englund, D.A.: An algorithm for determining $R(N)$ from the subscripts of the Zeckendorf representation of N , The Fibonacci Quarterly, Vol.39.3,pp250-52,(1999)
- (9) Horadam, A.F.: Zeckendorf representations of positive and negative integers by Pell numbers, Applications of Fibonacci Numbers, Vol.5,pp305-316, Kluwer Academic Publishers,(1993)
- (10) Horadam, A.F.: Maximal representation of positive integers by Pell numbers, The Fibonacci Quarterly, Vol.32.3, pp240-44,(1994)