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Some Fibonacci & Lucas identities via the Chebyshev polynomials

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Abstract

There exists a deep relationship between the Chebyshev polynomials and the Fibonacci & Lucas numbers. In this short note, I will show some new Fibonacci & Lucas identities via the Chebyshev polynomials.

1. Introduction

It is well-known that there are close relationships between the Chebyshev polynomials and the Fibonacci & Lucas numbers. In an earlier paper [6], written in Japanese, I revealed some of these relations, and got some Fibonacci & Lucas identities.

In this note, I will first summarize some of main results of my Japanese paper, and then I will show further Fibonacci & Lucas identities via the Chebyshev polynomials.

Definition The Chebyshev polynomials of the 1st & 2nd kinds are defined respectively by

$$T_n(x) = \cos n\theta, \quad U_n(x) = \sin(n+1)\theta / \sin\theta,$$

where $x = \cos\theta$. Note that once we get polynomial expressions of these functions, x varies in \mathbf{C} , the field of all the complex numbers.

In the following, I will use mostly the "modified" Chebyshev polynomials of the 1st & 2nd kinds:

$$t_n(x) = 2T_n(x/2), \quad u_n(x) = U_n(x/2),$$

which are monic polynomials of degree n ($n > 0$).

Note that these polynomials are the same as the "Vieta polynomials" (cf. N. Robbins[8]).

The following recurrent relations provide the easiest way to calculate these polynomials:

$$t_{n+2}(x) = xt_{n+1}(x) - t_n(x); \quad t_0(x) = 2, \quad t_1(x) = x. \quad \dots\dots (A1)$$

$$u_{n+2}(x) = xu_{n+1}(x) - u_n(x); \quad u_0(x) = 1, \quad u_1(x) = x. \quad \dots\dots (A2)$$

Here, (A1) corresponds to (1) in [6]. Similarly, the Theorem A1 & Proposition A1 correspond to the Theorem 1 & Proposition 1 in [6], respectively, and so on.

From the definition, the Chebyshev polynomials can be expressed by

$$t_n(x) = \left(\frac{x + \sqrt{x^2 - 4}}{2}\right)^n + \left(\frac{x - \sqrt{x^2 - 4}}{2}\right)^n \quad \dots\dots (A3)$$

$$u_n(x) = \left\{ \left(\frac{x + \sqrt{x^2 - 4}}{2}\right)^{n+1} - \left(\frac{x - \sqrt{x^2 - 4}}{2}\right)^{n+1} \right\} / \sqrt{x^2 - 4} \quad \dots\dots (A4)$$

Two kinds of these polynomials are related each other by the following equations:

$$t_n(x) = u_n(x) - u_{n-2}(x), \quad \dots\dots (A7)$$

$$(x^2 - 4)u_n(x) = t_{n+2}(x) - t_n(x), \quad \dots\dots (A8)$$

which are readily obtained from trigonometric identities:

$$\sin(n+1)\theta - \sin(n-1)\theta = 2 \sin \theta \cdot \cos n\theta,$$

and $\cos(n+2)\theta - \cos n\theta = -2 \sin \theta \cdot \sin(n+1)\theta.$

From these identities and some trigonometric identities, we can easily get the following Propositions:

Proposition A1

- (a) $\sum_{k=1}^n t_{2k-1}(x) = u_{2n-1}(x),$
- (b) $\sum_{k=1}^n t_{2k}(x) = u_{2n}(x) - 1,$
- (c) $\sum_{k=1}^n t_k(x) = u_n(x) + u_{n-1}(x) - 1,$
- (d) $(x^2 - 4) \sum_{k=1}^n u_{2k-1}(x) = t_{2n+1}(x) - x,$
- (e) $(x^2 - 4) \sum_{k=1}^n u_{2k}(x) = t_{2n+2}(x) - x^2 + 2,$
- (f) $(x^2 - 4) \sum_{k=1}^n u_k(x) = t_{n+2}(x) + t_{n+1}(x) - x^2 - x + 2,$
- (a') $\sum_{k=1}^n (-1)^{k-1} t_{2k-1}(x) = \frac{(-1)^{n-1} t_{2n}(x) + 2}{x},$
- (b') $\sum_{k=1}^n (-1)^{k-1} t_{2k}(x) = \frac{(-1)^{n-1} t_{2n+1}(x) + x}{x},$
- (c') $\sum_{k=1}^n (-1)^{k-1} t_k(x) = (-1)^{n-1} (u_n(x) - u_{n-1}(x)) + 1,$
- (d') $\sum_{k=1}^n (-1)^{k-1} u_{2k-1}(x) = \frac{(-1)^{n-1} u_{2n}(x) + 1}{x},$
- (e') $\sum_{k=1}^n (-1)^{k-1} u_{2k}(x) = \frac{(-1)^{n-1} u_{2n+1}(x) + x}{x},$
- (f') $\sum_{k=1}^n (-1)^{k-1} u_k(x) = (-1)^{n-1} (t_{n+2}(x) - t_{n+1}(x)) - x + 2.$

Proposition A2

- (a) $t_n(x)^2 - (x^2 - 4)u_{n-1}(x)^2 = 4,$
- (b) $t_{2n-1}(x) = t_n(x)t_{n-1}(x) - x,$
- (c) $t_{2n}(x) = t_n(x)^2 - 2,$
- (d) $u_{2n-1}(x) = t_n(x)u_{n-1}(x),$
- (e) $u_{2n}(x) = t_n(x)u_n(x) - 1,$
- (f) $t_{n+k}(x)t_{n-k}(x) - t_n(x)^2 = t_k(x)^2 - 4,$
- (g) $u_{n+k}(x)u_{n-k}(x) - u_n(x)^2 = -u_{k-1}(x)^2.$

Derivatives of the Chebyshev polynomials are as follows:

$$t'_n(x) = nu_{n-1}(x), \quad \dots\dots(A11)$$

$$(x^2 - 4)u'_n(x) = (n+1)t_{n+1}(x) - xu_n(x). \quad \dots\dots(A12)$$

Then the following Proposition is an easy consequence of the Proposition A1.

Proposition A4

- (a) $\sum_{k=1}^n kt_{2k-1}(x) = nu_{2n-1}(x) - \frac{t_{2n-1}(x) - x}{x^2 - 4}$,
- (b) $\sum_{k=1}^n kt_{2k}(x) = nu_{2n}(x) - u_{n-1}(x)^2$,
- (c) $\sum_{k=1}^n kt_k(x) = \frac{nt_{n+1}(x) - (n+1)t_n(x) + 2}{x - 2}$,
- (d) $(x^2 - 4) \sum_{k=1}^n ku_{2k-1}(x) = \frac{(2n+1)t_{2n+1}(x) - xu_{2n}(x)}{2}$,
- (e) $(x^2 - 4) \sum_{k=1}^n ku_{2k}(x) = nxu_{2n+1}(x) - (2n+1)u_{2n}(x) + 1$,
- (f) $(x^2 - 4) \sum_{k=1}^n ku_k(x) = (x+2) \{nu_{n+1}(x) - (n+1)u_n(x) + 1\}$,
- (a') $\sum_{k=1}^n (-1)^{k-1} kt_{2k-1}(x) = \frac{(-1)^{n-1} (nxt_{2n}(x) + t_{2n-1}(x)) + x}{x^2}$,
- (b') $\sum_{k=1}^n (-1)^{k-1} kt_{2k}(x) = \frac{(-1)^{n-1} (nxt_{2n+1}(x) + t_{2n}(x)) + 2}{x^2}$,
- (c') $\sum_{k=1}^n (-1)^{k-1} kt_k(x) = n(-1)^{n-1} (u_n(x) - u_{n-1}(x)) + \frac{(-1)^{n-1} t_n(x) + 2}{x + 2}$,
- (d') $\sum_{k=1}^n (-1)^{k-1} ku_{2k-1}(x) = \frac{(-1)^{n-1} (nxu_{2n}(x) + u_{2n-1}(x))}{x^2}$,
- (e') $\sum_{k=1}^n (-1)^{k-1} ku_{2k}(x) = \frac{(-1)^{n-1} (nxu_{2n+1}(x) + u_{2n}(x)) + 1}{x^2}$,
- (f') $\sum_{k=1}^n (-1)^{k-1} ku_k(x) = \frac{(-1)^{n-1} \{nu_{n+1}(x) + (n+1)u_n(x)\} + 1}{x + 2}$.

Now, $\cos n\theta$ is zero at $n\theta = (2k-1)\pi/2$, $\theta = (2k-1)\pi/2n$, or $\cos \theta = \cos \{(2k-1)\pi/(2n)\}$, ($k=1, 2, 3, \dots, n$).

It readily turns to a Chebyshev relation:

$$t_n(x) = 0 \Leftrightarrow x = 2 \cos \frac{2k-1}{2n} \pi, \quad (k=1, 2, \dots, n).$$

Since $\cos x$ is decreasing in $[0, \pi]$, all of these values are different. So we can factorize $t_n(x)$, as follows:

$$t_n(x) = \prod_{k=1}^n (x - 2 \cos \frac{2k-1}{2n} \pi).$$

After some routine transformations, we have the following Proposition:

Proposition A5

- (a) $t_n(x) = x^{n-2\lfloor n/2 \rfloor} \cdot \prod_{k=1}^{\lfloor n/2 \rfloor} (x^2 - 2 \cos \frac{2k-1}{n} \pi - 2)$,
- (b) $u_{n-1}(x) = x^{n-1-2\lfloor (n-1)/2 \rfloor} \cdot \prod_{k=1}^{\lfloor (n-1)/2 \rfloor} (x^2 - 2 \cos \frac{2k}{n} \pi - 2)$.

2. Some of main results of our earlier paper

It is well-known that the Fibonacci & Lucas numbers can be expressed by the forms

$$F_n = (\alpha^n - \beta^n) / \sqrt{5}, \tag{A15}$$

and $L_n = \alpha^n + \beta^n$(A16)

where $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$. Then we can get easily the following identities:

$$\alpha^n = \frac{L_n + \sqrt{5}F_n}{2}, \quad \beta^n = \frac{L_n - \sqrt{5}F_n}{2} \quad \dots\dots (A22)$$

and then

$$F_{mn} = \frac{1}{\sqrt{5}} \left\{ \left(\frac{L_n + \sqrt{5}F_n}{2} \right)^m - \left(\frac{L_n - \sqrt{5}F_n}{2} \right)^m \right\} \quad \dots\dots (A15)$$

$$L_{mn} = \left(\frac{L_n + \sqrt{5}F_n}{2} \right)^m + \left(\frac{L_n - \sqrt{5}F_n}{2} \right)^m \quad \dots\dots (A16)$$

All of these formulae are well-known (cf. Kelisky[5] and Castellanos[3]). Comparing these identities with (A3) & (A4), we can conclude that

$$\text{if } x = \begin{cases} \sqrt{5}F_n \text{ or } iL_n & (n:\text{odd}), \\ i\sqrt{5}F_n \text{ or } L_n & (n:\text{even}), \end{cases} \quad (\text{where } i = \sqrt{-1})$$

then the Chebyshev polynomials express (simple ratios of) the Fibonacci & Lucas numbers. All but 2 cases are got by Castellanos[3]. He missed (d) and (j) of the following Theorem.

Theorem A1 For any non-negative integer m and n , the following equalities hold:

- (a) $t_{2n-1}(\sqrt{5}F_{2m-1}) = \sqrt{5}F_{(2m-1)(2n-1)}$,
- (b) $t_{2n-1}(i\sqrt{5}F_{2m}) = i\sqrt{5}(-1)^{n-1}F_{2m(2n-1)}$,
- (c) $t_{2n}(\sqrt{5}F_{2m-1}) = L_{(2m-1) \cdot 2n}$,
- (d) $t_{2n}(i\sqrt{5}F_{2m}) = (-1)^n L_{4mn}$,
- (e) $t_n(L_{2m}) = L_{2mn}$,
- (f) $t_n(iL_{2m-1}) = i^n L_{(2m-1) \cdot n}$,
- (g) $u_{2n-1}(\sqrt{5}F_{2m-1}) = \sqrt{5}F_{(2m-1) \cdot 2n} / L_{2m-1}$,
- (h) $u_{2n-1}(i\sqrt{5}F_{2m}) = i\sqrt{5}(-1)^{n+1}F_{4mn} / L_{2m}$,
- (i) $u_{2n}(\sqrt{5}F_{2m-1}) = L_{(2m-1)(2n+1)} / L_{2m-1}$,
- (j) $u_{2n}(i\sqrt{5}F_{2m}) = (-1)^n L_{2m(2n+1)} / L_{2m}$,
- (k) $u_{n-1}(L_{2m}) = F_{2mn} / F_{2m}$,
- (l) $u_{n-1}(iL_{2m-1}) = i^{n-1} F_{(2m-1)n} / F_{2m-1}$.

Note that (e) & (f) are found in Kelisky[5]. For example, let $m=0$ in (f) & (l), then we have

$$t_n(i) = i^n L_n, \quad u_{n-1}(i) = i^{n-1} \cdot F_n. \quad \dots\dots (A23)$$

Let $m=0$ in (f) & (k), then we have

$$t_n(3) = L_{2n}, \quad u_{n-1}(3) = F_{2n}.$$

(Bernstein [2] called the latter identity "new formula". Cf. Rivlin[7], 1. 5. 57(d).) And so on.

Combining this theorem with the former Propositions, we can readily get the following Theorems:

Theorem A2 For any positive integers m and n , we have

- (a) $\sum_{k=1}^n F_{(2m-1)k} = \frac{F_{(2m-1)(n+1)} + F_{(2m-1)n} - F_{2m-1}}{L_{2m-1}}$,
- (b) $\sum_{k=1}^n F_{2mk} = \frac{L_{2m(n+1)} + L_{2mn} - L_{2m} - 2}{5F_{2m}}$,

$$\begin{aligned}
(c) \quad \sum_{k=1}^n L_{(2m-1)k} &= \frac{L_{(2m-1)(n+1)} + L_{(2m-1)n} - L_{2m} - 2}{L_{2m}}, \\
(d) \quad \sum_{k=1}^n L_{2mk} &= \frac{F_{2m(n+1)} + F_{2mn} - F_{2m}}{F_{2m}}, \\
(a') \quad \sum_{k=1}^n (-1)^{k-1} F_{(2m-1)k} &= \frac{(-1)^{n-1} (F_{(m-1)(n+1)} - F_{(2m-1)n}) + F_{2m-1}}{L_{2m-1}}, \\
(b') \quad \sum_{k=1}^n (-1)^{k-1} F_{2mk} &= \frac{(-1)^{n-1} (L_{2m(n+1)} - L_{2mn}) + L_{2m} - 2}{5F_{2m}}, \\
(c') \quad \sum_{k=1}^n (-1)^{k-1} L_{(2m-1)k} &= \frac{(-1)^{n-1} (L_{(2m-1)(n+1)} - L_{(2m-1)n}) + L_{2m-1} - 2}{L_{2m-1}}, \\
(d') \quad \sum_{k=1}^n (-1)^{k-1} L_{2mk} &= \frac{(-1)^{n-1} (F_{2m(n+1)} - F_{2mn}) + F_{2m}}{F_{2m}}.
\end{aligned}$$

Note that (a) & (b) are found in Castellanos[3].

Theorem A4

$$\begin{aligned}
(a) \quad \sum_{k=1}^n kF_{2k-1} &= nF_{2n} - F_{2n-1} + 1, \\
(b) \quad \sum_{k=1}^n kF_{2k} &= nF_{2n+1} - F_{2n}, \\
(c) \quad \sum_{k=1}^n kF_k &= (n-1)F_{n+2} - F_{n+1} + 2, \\
(d) \quad \sum_{k=1}^n kL_{2k-1} &= nL_{2n} - L_{2n-1} - 1, \\
(e) \quad \sum_{k=1}^n kL_{2k} &= nL_{2n+1} - L_{2n} + 2, \\
(f) \quad \sum_{k=1}^n kL_k &= (n-1)L_{n+2} - L_{n+1} + 4, \\
(a') \quad \sum_{k=1}^n (-1)^{k-1} kF_{2k-1} &= \frac{(-1)^{n-1} \{nF_{2n} + (2n+1)F_{2n-1}\} + 1}{5}, \\
(b') \quad \sum_{k=1}^n (-1)^{k-1} kF_{2k} &= \frac{(-1)^{n-1} \{(2n+1)L_{2n+1} - F_{2n+1}\}}{10}, \\
(c') \quad \sum_{k=1}^n (-1)^{k-1} kF_k &= (-1)^{n-1} \{(n+1)F_{n-1} - F_n\} + 2, \\
(d') \quad \sum_{k=1}^n (-1)^{k-1} kL_{2k-1} &= \frac{(-1)^{n-1} (5nF_{2n} + L_{2n-1}) - 1}{5}, \\
(e') \quad \sum_{k=1}^n (-1)^{k-1} kL_{2k} &= \frac{(-1)^{n-1} (5nF_{2n+1} + L_{2n}) + 2}{5}, \\
(f') \quad \sum_{k=1}^n (-1)^{k-1} kL_k &= (-1)^{n-1} \{(n+2)L_{n-1} - L_n\} - 4.
\end{aligned}$$

Note that some of these identities are the same as Fibonacci-Lucas identities obtained by G. Wulczyn[10](in slightly different forms).

As a bonus, we can get nice formulae expressing the Fibonacci & Lucas numbers:

Theorem A5

$$\begin{aligned}
(a) \quad F_n &= \prod_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(3 + 2 \cos \frac{2k\pi}{n} \right), \\
(b) \quad L_n &= \prod_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(3 + 2 \cos \frac{(2k-1)\pi}{n} \right),
\end{aligned}$$

$$(c) \quad F_{mn} = F_m \cdot L_m^{n-1} \cdot 2^{1-\frac{n-1}{2}} \prod_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (L_{2m} + 2(-1)^{m-1} \cos \frac{2k\pi}{n}),$$

$$(d) \quad L_{mn} = L_m^{n-2} \cdot 2^{\lfloor \frac{n}{2} \rfloor} \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} (L_{2m} + 2(-1)^{m-1} \cos \frac{(n-k)\pi}{n}).$$

Note that (a) is solved repeatedly. See Bedrosian [1] and solution of advanced problem (H-93) in the Fibonacci Quarterly(cf. (H-111) and (H-466), see also (H-64)).

3. Further results of similar type

We know that a trigonometric identity leads a Chebyshev identity, and then it leads a number of the Fibonacci & Lucas identities.

For example, a simple trigonometric identity

$$\sin(2n+1)\theta - \sin \theta = 2 \sin n\theta \cdot \cos(n+1)\theta$$

turns to a Chebyshev identity

$$u_{2n}(x) - 1 = u_{n-1}(x)t_{n+1}(x)$$

and then, it turns (by letting $x=L_{2m}$) to a Fibonacci/Lucas identity:

$$F_{2m(2n+1)} - F_{2m} = F_{2mn} \cdot L_{2m(n+1)}. \tag{1}$$

Similarly, starting from a trigonometric identity

$$\sin(n+k)\theta \cdot \cos m\theta - \sin n\theta \cdot \cos(m+k)\theta = \sin k\theta \cdot \cos(n-m)\theta$$

we get in turn

$$u_{n+k-1}(x)t_m(x) - u_{n-1}(x)t_{m+k}(x) = u_{k-1}(x)t_{n-m}(x)$$

$$\text{and} \quad F_{n+k}L_m - F_nL_{m+k} = (-1)^m F_k L_{n-m}. \tag{2}$$

This is a generalization of (I28) of Hoggatt[4]. In this fashion, we can get the following:

Theorem 1

- (a) $\sum_{k=0}^n \binom{n}{k} F_{2k} = \begin{cases} 5^{(n-1)/2} \cdot L_n, & (n:\text{odd}) \\ 5^{n/2} \cdot F_n, & (n:\text{even}) \end{cases}$
- (b) $\sum_{k=0}^n \binom{n}{k} L_{2k} = \begin{cases} 5^{(n+1)/2} \cdot F_n, & (n:\text{odd}) \\ 5^{n/2} \cdot L_n, & (n:\text{even}) \end{cases}$
- (c) $\sum_{k=0}^n \binom{n}{k} F_{4k} = 3^n \cdot F_{2n},$
- (d) $\sum_{k=0}^n \binom{n}{k} L_{4k} = 3^n \cdot L_{2n},$
- (e) $\sum_{k=0}^n \binom{n}{k} F_{(4m+2)k} = \begin{cases} 5^{(n-1)/2} \cdot F_{2m+1}^n L_{(2m+1)n}, & (n:\text{odd}) \\ 5^{n/2} \cdot F_{2m+1}^n F_{(2m+1)n}, & (n:\text{even}) \end{cases}$
- (f) $\sum_{k=0}^n \binom{n}{k} L_{(4m+2)k} = \begin{cases} 5^{(n+1)/2} \cdot F_{2m+1}^n F_{(2m+1)n}, & (n:\text{odd}) \\ 5^{n/2} \cdot F_{2m+1}^n L_{(2m+1)n}, & (n:\text{even}) \end{cases}$
- (g) $\sum_{k=0}^n \binom{n}{k} F_{4mk} = L_{2m}^n \cdot F_{2mn},$
- (h) $\sum_{k=0}^n \binom{n}{k} L_{4mk} = L_{2m}^n \cdot L_{2mn}.$

(proof) From Euler's formula, it's easy to get a trigonometric identity

$$\begin{aligned} (1 + \cos \theta + i \sin \theta)^n &= (1 + e^{i\theta})^n = \sum_{k=0}^n \binom{n}{k} e^{i k \theta} \\ &= \sum_{k=0}^n \binom{n}{k} \cos k \theta + i \sum_{k=0}^n \binom{n}{k} \sin k \theta \end{aligned} \quad \dots\dots (3)$$

which, in turn, leads a Chebyshev identity:

$$\begin{aligned} (x + 2 + \sqrt{x^2 - 4})^n &= 2^{n-1} \left\{ \sum_{k=0}^n \binom{n}{k} t_k(x) + \sqrt{x^2 - 4} \sum_{k=0}^n \binom{n}{k} u_{k-1}(x) \right\}. \end{aligned} \quad \dots\dots (4)$$

Let $x=3$ in (4), and we have

$$2^{n-1} \left(\sum_{k=0}^n \binom{n}{k} L_{2k} + \sqrt{5} \sum_{k=0}^n \binom{n}{k} F_{2k} \right) = (5 + \sqrt{5})^n$$

that is

$$\sum_{k=0}^n \binom{n}{k} L_{2k} + \sqrt{5} \sum_{k=0}^n \binom{n}{k} F_{2k} = 2\sqrt{5}^n \alpha^n = \sqrt{5}^n (L_n + \sqrt{5}F_n).$$

Thus we get (a) & (b). Similarly, let $x=7$ (& L_{2m}), and we have (c) & (d) (and (e), (f), (g) & (h), respectively).

Note that (a) & (b) are (69) & (70) of Vajda[9], (appendix;list of formulae). In the sequel, we shall write (V69) & (V70). □

Starting from identities

$$\begin{aligned} (-1 + \cos \theta + i \sin \theta)^n &= (-1 + e^{i\theta})^n \\ &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \cos k \theta + i \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \sin k \theta \end{aligned} \quad \dots\dots (5)$$

and

$$\begin{aligned} (x - 2 + \sqrt{x^2 - 4})^n &= 2^{n-1} \left\{ \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} t_k(x) + \sqrt{x^2 - 4} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} u_{k-1}(x) \right\} \end{aligned} \quad \dots\dots (6)$$

we get the following:

Theorem 2

- (a) $\sum_{k=0}^n (-1)^k \binom{n}{k} F_{2k} = (-1)^n F_n,$
- (b) $\sum_{k=0}^n (-1)^k \binom{n}{k} L_{2k} = (-1)^n L_n,$
- (c) $\sum_{k=0}^n (-1)^{k-1} \binom{n}{k} F_{4k} = \begin{cases} 5^{(n-1)/2} L_{2n}, & (n:\text{odd}) \\ -5^{n/2} F_{2n}, & (n:\text{even}) \end{cases}$
- (d) $\sum_{k=0}^n (-1)^{k-1} \binom{n}{k} L_{4k} = \begin{cases} 5^{(n+1)/2} F_{2n}, & (n:\text{odd}) \\ -5^{n/2} L_{2n}, & (n:\text{even}) \end{cases}$
- (e) $\sum_{k=0}^n (-1)^{k-1} \binom{n}{k} F_{(4m+2)k} = (-1)^{n-1} L_{2m+1}^n F_{(2m+1)n},$

$$\begin{aligned}
(e') \quad \sum_{k=0}^n (-1)^{k-1} \binom{n}{k} F_{4mk} &= \begin{cases} 5^{(n-1)/2} \cdot F_{nm} L_{2mn}, & (n:\text{odd}) \\ -5^{n/2} \cdot F_{2m} F_{2mn}, & (n:\text{even}) \end{cases} \\
(f) \quad \sum_{k=0}^n (-1)^{k-1} \binom{n}{k} L_{(4m+2)k} &= (-1)^{n-1} L_{nm+1} L_{(2m+1)n}, \\
(f') \quad \sum_{k=0}^n (-1)^{k-1} \binom{n}{k} L_{4mk} &= \begin{cases} 5^{(n+1)/2} \cdot F_{2m} F_{2mn}, & (n:\text{odd}) \\ -5^{n/2} \cdot F_{2m} L_{2mn}, & (n:\text{even}) \end{cases}
\end{aligned}$$

Note that (a) & (b) are (V71) & (V72).

Combining Euler's formula and the sum formula of a geometric series, we have, for any real number p ,

$$\begin{aligned}
\sum_{k=0}^n p^k \cos k\theta + i \sum_{k=0}^n p^k \sin k\theta &= \sum_{k=0}^n (pe^{i\theta})^k \\
&= \frac{p^{n+1} \{\cos(n+1)\theta + i \sin(n+1)\theta\} - 1}{p(\cos\theta + i \sin\theta) - 1} \quad \dots\dots(7)
\end{aligned}$$

$$\begin{aligned}
\therefore \sum_{k=0}^n p^k t_k(x) + \sqrt{x^2-4} \sum_{k=0}^n p^k u_{k-1}(x) \\
&= \frac{p^{n+1} \{t_{n+1}(x) + \sqrt{x^2-4} u_n(x)\} - 4}{px - 2 + p\sqrt{x^2-4}} \quad \dots\dots(8)
\end{aligned}$$

Thus we get the following Theorems.

Theorem 3

$$\begin{aligned}
(a) \quad \sum_{k=0}^n p^k F_k &= \frac{p^{n+1} (F_{n+1} + pF_n) - p}{p^2 + p - 1}, \\
(b) \quad \sum_{k=0}^n p^k L_k &= \frac{p^{n+1} (L_{n+1} + pL_n) + p - 2}{p^2 + p - 1}, \\
(c) \quad \sum_{k=0}^n p^k F_{2k} &= \frac{p^{n+1} \{(p-2)F_{2n} - F_{2n-1}\} + p}{p^2 - 3p + 1}, \\
(d) \quad \sum_{k=0}^n p^k L_{2k} &= \frac{p^{n+1} \{(p-2)L_{2n} - L_{2n-1}\} - 3p + 2}{p^2 - 3p + 1},
\end{aligned}$$

(proof) Let $x = \sqrt{5}$ and 3 in (8), then the Theorem follows immediately.

Note that (a) is obtained by C. Podilla as a solution to the problem B98. \square

Next, we shall extend, only a little bit, the Theorem A4.

Theorem 4

For all positive integers m and n ,

$$\begin{aligned}
(a) \quad F_{2m}^3 \sum_{k=1}^n k^2 L_{2m(2k+1)} &= n^2 F_{2m} F_{4m(n+1)} - 2n F_{2m} L_{2m(2n+1)} + L_{2m} F_{2mn} L_{2m(n+1)} \\
(b) \quad 5 \cdot F_{2m}^3 \sum_{k=1}^n k^2 L_{4mk} &= (5n^2 F_{2m}^2 + 2) F_{2m(2n+1)} - (2n+1) F_{2m} L_{4mn}, \\
(c) \quad L_{2m}^3 \sum_{k=1}^n (-1)^{k-1} k^2 L_{2m(2k+1)} &= (-1)^{n-1} \{n(n+1) L_{2m}^2 L_{4m(n+1)} - 5n F_{2m} L_{2m} F_{4m(n+1)} \\
&\quad - 5F_{2m} F_{2m(2n+1)}\} - 5F_{2m}^2 \\
(d) \quad L_{2m}^3 \sum_{k=1}^n (-1)^{k-1} k^2 L_{4mk} &= (-1)^{n-1} \{5(2n+1) F_{2m} F_{4mn} - n(nL_{2m}^2 + 4) L_{2m(2n+1)}\}
\end{aligned}$$

(proof) These are readily obtained by the Chebyshev identities:

$$x^2 \sum_{k=1}^n (-1)^{k-1} k^2 u_{2k}(x) = (-1)^{n-1} \{n^2 x u_{2n+1}(x) + 2n u_{2n}(x) - \frac{1}{x} t_{2n+1}(x)\} - 1 \quad \dots\dots (9)$$

$$x^3 \sum_{k=1}^n (-1)^{k-1} k^2 t_{2k}(x) = (-1)^{n-1} \{(n^2 x^2 - 2) t_{2n+1}(x) + (2n+1) x t_{2n}(x)\} \quad \dots\dots (10)$$

$$(x^2 - 4) \sum_{k=1}^n k^2 u_{2k}(x) = n(n+1) t_{2n+2}(x) - n x u_{2n+1}(x) + x(t_{2n+1}(x) - x) / (x^2 - 4) \quad \dots\dots (11)$$

$$(x^2 - 4) \sum_{k=1}^n k^2 t_{2k}(x) = (2n+1) x u_{2n-1}(x) + n(n x^2 - 4 - 4n) u_{2n}(x) \quad \dots\dots (12)$$

We only have to apply (d) & (j) of Theorem A1. □

References

- [1] Bedrosian, S. D. :The Fibonacci numbers via trigonometric expressions, **J. Franklin Inst.**, **295**, pp. 175-177(1973)
- [2] Bernstein, L. :A formula for Fibonacci numbers from a new approach to generalized Fibonacci numbers, **Fibonacci Quart.**, **14**, pp. 358-368(1976)
- [3] Castellanos, D. :A generalization of Binet's formula and some of its consequences; **Fibonacci Quart.**, **27**, pp. 424-438(1989)
- [4] Hoggatt, V. E. Jr. : **Fibonacci and Lucas numbers**, Fibonacci Association, (1969)
- [5] Kelisky, R. P. :On formulas involving both the Bernoulli and Fibonacci numbers; **Scripta Math.**, **23**, pp. 27-35(1958)
- [6] Nakamura, S. :On the Chebyshev polynomials and the Fibonacci and Lucas numbers (Japanese, with English Summary); **J. Tokyo Univ. of Mercantile Marine** (Natural Sci.), **42**, pp. 101-116(1992)
- [7] Rivlin, T. J. : **Chebyshev polynomials**, 2nd ed., Wiley-Interscience(1990)
- [8] Robbins, N. :Vieta's triangular array and a related family of polynomials, **Internat. J. Math. & Math. Sci.**, **14**, pp. 239-244(1991)
- [9] Vajda, S. : **Fibonacci & Lucas numbers, and the golden section: theory and application**, Ellis Horwood (John Wiley), (1989)
- [10] Wulczyn, G. : Formation of generalized F-L identities of the form $\sum_{r=1}^n r^s \bar{F}_{h r+a}$, **A collection of manuscripts related to the Fibonacci sequence**, (ed. by V.E.Hoggatt & M.Bicknell-Johnson), The Fibonacci Association, pp. 192-201(1980)

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