自然数Nのベル表現の個数についてⅡ

On the Number of Pellian Representations of Natural Number N II

OSAMU MATSUSHITA

Abstract

Pell sequence is defined by the second order linear recurece such that $P_{n+1} = 2P_n + P_{n-1}$ with $P_0 = 1, P_1 = 1$ and its terms are called Pell numbers. It is known that any posive integer N can be represented by the sum of Pell numbers as follows: $N = \sum_{i=1}^{n} \alpha_i P_i$ where $0 \leq \alpha_i \leq 2$. Such a representation is called a Pellian representation of N.

We denote the number of Pellian representations of N by $R(N)$. The purpose of this paper are to show some properties of $R(N)$ and the method of computing $R(N)$ by using those properties. We set $\sum_{k=1}^{n} P_k = S_n$. Main results are as follows;

(1) $R(S_n) = 1$

(2) $R(P_n) = \left[\dfrac{n+2}{2}\right]$

(3) $R(S_n + 1) = R(S_n - 1) = 2$

(4) $0 \leq M \leq \left[\dfrac{P_{n+1}}{2}\right] \Longrightarrow R(S_n + M) = R(S_{n+1} - M)$

(5) $S_n < S_n + M < P_{n+1} \Longrightarrow R(S_n + M) = R(S_n + M - P_n)$

(6) $P_{n+1} < S_n + M < P_{n+1} + P_n \Longrightarrow R(S_n + M) = R(S_n + M - P_{n+1}) + R(S_n + M - 2P_n)$

(7) $P_{n+1} < S_n + M \leq S_n + \left[\dfrac{P_{n+1}}{2}\right] \Longrightarrow R(S_n + M) = R(S_n + M - P_{n+1}) + R(S_n + M - 2P_n)$.

1 Pell numbers and Zeckedorf's theorem

Fibonacci seqeunce $\{F_n\}$ is defined by the second order linear recurence such that $F_{n+1} = F_n + F_{n-1}$ with $F_0 = 0$ and $F_1 = 1$, and the number appearing in the seqeunce is called a Fibonacci number. A great amount of theorems have been discoverd by many mathematicians since 13th century when Leonard Pisa,Fibonacci,described the Fibonacci number in his famous book 'Liber abati'.But Zeckendorf's theorem which we shall state is comparatively new.

平成14年9月10日受付

Theorem 1.1 (Zeckendorf[2][3]). *Any positive integer are represented uniquely as follows;*

$$N=\sum_{i=1}^{n}\alpha_i F_i \quad (\ \alpha_i=0 \ \ or\ 1,\ \alpha_i\alpha_{i+1}=0,\ \alpha_1=0\ )$$

This theorem asserts that any natural number N can be uniquely represented by using Fibonacci numbers with the condition such that $\alpha_i = 0$ or 1, $\alpha_i\alpha_{i+1} = 0$, $\alpha_1 = 0$. If we don't assume that $\alpha_i\alpha_{i+1} = 0$, the representation is not unique in general. Such a representations of N is called a Fibnonacci representation of N. We shall denote the numbers of the Fibonacci repressentations of N by $R_F(N)$. There have been a lot of investigations[4],[5],[6],[7],[8] of the property $R_F(N)$.

Definition 1.2. *Pell sequence $\{P_n\}$ is the sequence which is satisfied the linear recurrece relation such that $P_{n+1} = 2P_n + P_{n-1}$ with the condition $P_0 = 0$, $P_1 = 1$. Each P_n is a called Pell number.*

Zeckendorf theorem for Pell numbers is as follows.

Theorem 1.3. [9],[10] *Every positive integer has a unique representation in the form*

$$N=\sum_{i=1}^{n}\alpha_i P_i$$

where $\alpha_i = 0, 1$ or 2, $\alpha_i = 2 \Rightarrow \alpha_{i-1} = 0$.

Similarly to Fibonacci numbers if we do not suppose the condition that $\alpha_i = 2 \Rightarrow \alpha_{i-1} = 0$,the representation is not unique in general.Such a represention is called a Pellian representation.

Henceforce we shall denote the number of representations of N by $R(N)$.

2 The property of $R(N)$

First of all we shall sum up the results in the author's previous papaer[1].

Definition 2.1. $S_n = P_1 + \cdots + P_n$.

Lemma 2.2. *For $n \geq 1$, we have* $S_n = \dfrac{P_{n+1} + P_n - 1}{2}$.

Theorem 2.3. $N = S_n \Longleftrightarrow R(N) = 1$.

Theorem 2.4. $0 \leq M \leq \left[\dfrac{P_{n+1}}{2}\right] \Longrightarrow R(S_n + M) = R(S_{n+1} - M)$.

Theorem 2.5. *For $n = 0, 1, 2, 3, \cdots$,* $R(P_n) = \left[\dfrac{n+2}{2}\right]$ $(n = 0, 1, 2, \cdots)$ *where* $R(0) = 1$.

The proof of these theorems can be seen in Matsushita[1].

Lemma 2.6. $S_n + S_{n-1} + 1 = P_{n+1}$

We can easily prove this lemma by using lemma 2.2.

Theorem 2.7. $S_n < S_n + M < P_{n+1} \Longrightarrow R(S_n + M) = R(S_n + M - P_n)$

Proof. The Pellian representation of $S_n + M$ contains P_n. This claim can be shown as follows. Since $P_n + M < P_{n+1}$,the Pellian representation of $S_n + M$ does not contain P_{n+1}.
Moreover from lemma 2.2,

$$\begin{aligned} 2(P_1 + P_2 + \cdots + P_{n-1}) &= P_n + P_{n-1} - 1 \\ &< S_n \\ &< S_n + M. \end{aligned}$$

Hence the Pellian representation of $S_n + M$ must contains P_n.
Therefore from the equation $S_n + M = P_n + (S_n + M - P_n)$,we completes the proof. □

Remark. Since $S_n - P_n = S_{n-1}$ and $P_{n+1} - S_n - 1 = S_{n-1}$,the above theorem can be written as follows;

$$0 \leq M \leq S_{n-1} \Longrightarrow R(S_n + M) = R(S_{n-1} + M)$$

From this remark we can prove the next theorem easily.

Theorem 2.8. $R(S_n + 1) = R(S_n - 1) = 2$.

Proof. By using the formula in the above remark repetedly、we obtain $R(S_n + 1) = R(S_1 + 1) = R(2) = 2$. On the other hand from theorem 2.4, we have $R(S_n - 1) = R(S_{n-1} + 1)$.
Hence, $R(S_n - 1) = 2$. □

When $P_{n+1} < S_n + M < P_n + P_{n+1}$, we have the following theorem.

Theorem 2.9. $P_{n+1} < S_n + M < P_{n+1} + P_n \Longrightarrow R(S_n + M) = R(S_n + M - P_{n+1}) + R(S_n + M - 2P_n)$.

To prove the above theorem we need the following lemmas .

Lemma 2.10. *When $P_{n+1} < N$, if P_{n+1} is not used in a Pellian representation of N. then P_n is used twice in the representation.*

Proof. Since $P_{n+1} < N$ and from lemma 2.2,

$$\begin{aligned} N - P_n > P_{n+1} - P_n &= P_n + P_{n-1} \\ &> 2(P_1 + P_2 + \cdots + P_{n-1}) \end{aligned}$$

Hence P_n must be used in the representation of $N - P_n$. □

Lemma 2.11. *Let $P_{n+1} < S_n + M < P_{n+1} + P_n$.*
$\Longrightarrow$
(1) If we use P_{n+1} for representing $S_n + M$, we don't use P_n.
(2) If we do not use P_{n+1} for representing $S_n + M$,we use $P_n$twice.

Proof. Since $S_n + M < P_{n+1} + P_n$,we have $S_n + M - P_{n+1} < P_n$. So (1) is obvious. (2) is already proved in lemma 2.10. □

From lemma 2.10 and lemma 2.2, we can easily prove the theorem 2.9.

Lemma 2.12. $S_n + \left[\frac{P_{n+1}}{2}\right] < P_{n+1} + P_n$

Proof. From lemma 2.2,

$$\begin{aligned} S_n + \left[\frac{P_{n+1}}{2}\right] &\leqq \frac{P_n + P_{n+1} - 1}{2} + \frac{P_{n+1}}{2} \\ &= P_{n+1} + \frac{P_n - 1}{2} \\ &< P_{n+1} + P_n. \end{aligned}$$

□

Corollay 2.13. $P_{n+1} < S_n + M \leq S_n + \left[\frac{P_{n+1}}{2}\right]$

$$\Longrightarrow R(S_n + M) = R(S_n + M - P_{n+1}) + R(S_n + M - 2P_n)$$

3 Computation of $R(N)$

The results we obtained in the previous section are as follows;

(1) $R(S_n) = 1$

(2) $R(P_n) = \left[\frac{n+2}{2}\right]$

(3) $R(S_n + 1) = R(S_n - 1) = 2$

(4) $0 \leq M \leq \left[\frac{P_{n+1}}{2}\right] \Longrightarrow R(S_n + M) = R(S_{n+1} - M)$

(5) $S_n < S_n + M < P_{n+1} \Longrightarrow R(S_n + M) = R(S_n + M - P_n)$

(6) $P_{n+1} < S_n + M < P_{n+1} + P_n \Longrightarrow R(S_n + M) = R(S_n + M - P_{n+1}) + R(S_n + M - 2P_n)$

(7) $P_{n+1} < S_n + M \leq S_n + \left[\frac{P_{n+1}}{2}\right] \Longrightarrow R(S_n + M) = R(S_n + M - P_{n+1}) + R(S_n + M - 2P_n)$.

Combining these rsults we can compute $R(N)$.
Below is the table of $P_n$and S_n for $1 \leq n \leq 10$.Now using this table we shall compute $R(N)$ for some natural numbers N.

n	1	2	3	4	5	6	7	8	9	10
P_n	1	2	5	12	29	70	169	408	985	2378
S_n	1	3	8	20	49	119	288	696	1681	4059

Example 3.1. $R(1000)$

$P_9 = 985 < 1000 = 696 + 304 = S_8 + 304$. So, we have from (5)

$$\begin{aligned} R(1000) &= R(1000 - 985) + R(1000 - 2 \cdot 408) \\ &= R(15) + R(184). \end{aligned}$$

Now we compute $R(15), R(184)$.
$R(15) = R(20 - 5) = R(8 + 5) = R(13 - 12) + R(13 - 10) = R(1) + R(3) = 1 + 1 = 2$
$R(184) = R(119 + 65) = R(184 - 169) + R(184 - 140) = R(15) + R(44) = 2 + R(44)$.
And $R(44) = R(49 - 5) = R(20 + 5) = R(20 - 5) = R(15) = 2$.
Finally we obtain $R(1000) = 6$.

Example 3.2. $R(2000)$.

$P_9 = 985 < 2000 = 1681 + 319 = S_9 + 319$. So we get from (5)

$$\begin{aligned} R(2000) &= R(1681 + 319) \\ &= R(696 + 319) \\ &= R(1015 - 985) + R(1015 - 2 \cdot 408) \\ &= R(30) + R(199). \end{aligned}$$

Now we compute
$R(199) = R(119 + 80) = R(199 - 169) + R(199 - 140) = R(30) + R(59)$
and

$R(59) = R(49+10) = R(20+10) = R(30)$.
Therefore, $R(2000) = 3\,R(30)$.
$R(30) = R(20+10) = R(30-29) + R(30-24) = R(1) + R(6)$.
And $R(6) = R(8-3) = R(5) = R(P_3) = \left[\dfrac{3+1}{2}\right] = 2$.
Hence $R(30) = 1 + 2 = 3$ and finally we have $R(2000) = 3 \cdot 3 = 9$.

Example 3.3. $R(800)$

$$\begin{aligned} R(800) &= R(696+104) = R(288+104) = R(119+104) \\ &= R(288-65) = R(119+65) = R(184-169) + R(184-140) \\ &= R(15) + R(44) = R(15) + R(15) = 4 \end{aligned}$$

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