ラプラシアンから楕円型作用素へ

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ラプラシアンから楕円型作用素へ

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（4年生）

服部先生と砂田先生の指導の下で次の本を読んだ.

Berger-Gauduchuon-Mazet,
Le Spectre d'une variété Riemannienne,
Lect Notes Math 194

（修士）

足立先生の指導の下で次の論文を読んだ.

Atiyah-Singer,
The index of elliptic operators I$\sim$V
Ann. of Math

Definition

Let (M, g) be an oriented closed Riemannian manifold.

$$(\omega, \eta) = \int_M \omega \wedge *\eta \quad \text{for } \omega,\ \eta \in \Omega^p \, (: \ p\text{-forms})$$

d^* is the formal adjoint of d defined by

$$(d\omega, \eta) = (\omega, d^*\eta) \, .$$

Then the Laplacian $\triangle$ is defined by

$$\triangle = d^*d \, : \, C^\infty(M) = \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^*} C^\infty(M) \, .$$

$$(\triangle = (d + d^*)^*(d + d^*) = d^*d + dd^* \ : \ \Omega^p \longrightarrow \Omega^p)$$

Definition

λ is called a spectrum of (M, g) iff

$$\exists f \neq 0 \in C^\infty(M)\,,\ \triangle f = \lambda f\,.$$

Spectra are real non-negative discrete numbers.

Example Spectra of (S^n, g_0) are

$$\lambda_\ell = \ell(n+\ell-1)\ (\ell \geq 0) \ \text{ and}$$

$$\begin{aligned} m_\ell &= \text{multiplicity of } \lambda_\ell \\ &= \frac{(n+\ell-2)!}{\ell!(n-1)!}(n+2\ell-1) \end{aligned}$$

(Sketch of the proof)

Let $\triangle^R$, $\triangle^S$ be the Laplacian of $\left(\mathbf{R}^{n+1}, g_0\right)$, (S^n, g_0) respectively.

$$
\begin{aligned}
P_\ell &= \left\{\text{homogeneous polynomials on } \mathbf{R}^{n+1} \text{ of degree } \ell\right\} \\
&\qquad \supset H_\ell = \left\{f \in P_\ell \,|\, \triangle^R f = 0\right\}, \\
P_\ell^S &= \{f|_{S^n} \in C^\infty(S^n) \,|\, f \in P_\ell\} \\
&\qquad \supset H_\ell^S = \{f|_{S^n} \in C^\infty(S^n) \,|\, f \in H_\ell\}.
\end{aligned}
$$

Then direct calculation shows that $\triangle^S f = \lambda_\ell f$ for $f \in H_\ell^S$ $\left(\dim H_\ell^S = m_\ell\right)$ and $(f, g) = 0$ if $\triangle^S f = \lambda f$, $\triangle^S g = \mu g$ $(\lambda \neq \mu)$. Here $V = \oplus_{\ell \geq 0} H_\ell^S$ coincides with $\oplus_{\ell \geq 0} P_\ell^S$ and V is shown to be dense in $C^\infty(S^n)$ by using a theorem of Stone-Weierstrass. Hence any eigenfunction of $\triangle^S$ is contained in V.

Stationary state of the electron in a hydrogen

$$\psi(t,r,\theta,\phi) = e^{-iEt/\hbar}\varphi(r,\theta,\phi) \Longrightarrow |\psi|^2 = |\varphi(r,\theta,\phi)|^2$$

$\varphi = \varphi(r,\theta,\phi)$ is a solution of the Schrödinger equation

$$\left\{\frac{\hbar^2}{2m}\triangle^R + V(r)\right\}\varphi = E\varphi \quad \cdots ①$$

where $V(r) = -\dfrac{\varepsilon}{r}$ is the Coulomb potential.

Set $\varphi(r,\theta,\phi) = R(r)Y(\theta,\phi)$. Then

$$
① \iff \begin{cases} \triangle^S Y(\theta,\phi) = \lambda Y(\theta,\phi) \quad \cdots ② \\ \\ -\dfrac{\hbar^2}{2m}\left(\dfrac{d^2R(r)}{dr^2} + \dfrac{2}{r}\dfrac{dR(r)}{dr} - \dfrac{\lambda}{r^2}R(r)\right) \\ \\ \qquad\qquad +V(r)R(r) - ER(r) = 0 \quad \cdots ③ \end{cases}
$$

where $\triangle^S = -\left(\dfrac{\partial^2}{\partial\theta^2} + \dfrac{\cos\theta}{\sin\theta}\dfrac{\partial}{\partial\theta} + \dfrac{1}{\sin^2\theta}\dfrac{\partial^2}{\partial\phi^2}\right)$

is the Laplacian for (S^2, g_0) .

Then $R(r)$ must satisfy the following condition:

$$
\int_{\mathbf{R}^3} V(r)\,|\psi|^2 < \infty \iff \int_0^\infty rR(r)^2\,dr < \infty \quad \cdots ④
$$

$$② \Longrightarrow \lambda = \lambda_\ell = \ell(\ell+1) \ (\ell \in \mathbf{N} \cup \{0\}) \ \cdots ⑤$$

$$(\text{multiplicity of } \lambda_\ell = 2\ell + 1)$$

$$③, ④, ⑤ \Longrightarrow \begin{cases} R(r) = C r^\ell e^{-ar} \times \mathrm{poly(r)} \\ E = E_\ell = -\dfrac{m\varepsilon^2}{2\hbar^2} \dfrac{1}{n^2} \ (n \geq \ell + 1) \end{cases}$$

The difference $E_n - E_{\ell+1}$ of the energy leads to the spectra in the sunlight.

For example, the difference $E_n - E_2$ $(n \geq 3)$ leads to the Balmer series in the visible rays of the sun.

Notice If the solution $R(r)$ for a spectrum λ of (S^2, g_0) is smooth at $r = 0$, then λ is equal to $\ell(\ell + 1)$ for some nonnegative integer ℓ.

(Because)

Since λ is nonnegative, there uniquely exists a nonnegative real number η such that $\lambda = \eta(\eta + 1)$. Then the solution of ③, ④ is expressed as

$$R(r) = Cr^{\,\eta} e^{-ar} \times \mathrm{poly}(r)\,,$$

which is smooth at $r = 0$ if and only if η is a nonnegative integer. $\square$

Equivariant determinant of elliptic operators

Definition For a compact Lie group G and a G-equivariant elliptic operator D, a homomorphism $I_D : G \to \mathbf{R}/\mathbf{Z}$ is defined by

$$I_D(g) = \frac{1}{2\pi\sqrt{-1}} \log \frac{\det(g|\ker D)}{\det(g|\operatorname{coker} D)} .$$

Formula If $g \in G$ has an order p, then the next equality holds.

$$I_D(g) \equiv \frac{p-1}{2p}\mathrm{Index}(D) - \frac{1}{p}\sum_{k=1}^{p-1} \frac{1}{1-\xi_p^{-k}}\mathrm{Index}(D, g^k) \ \mathrm{mod.}\mathbf{Z}$$

where $\xi_p = e^{2\pi\sqrt{-1}/p}$ and

$$\mathrm{Index}(D, g) = \mathrm{Tr}\,(g|\ker D) - \mathrm{Tr}\,(g|\operatorname{coker} D)\,,$$

$$\mathrm{Index}(D) = \mathrm{Index}(D, 1) = \dim\ker D - \dim\operatorname{coker} D$$

Properties

(1) I_D is an additive homomorphism, and
hence the the following equalities hold:
$I_D(g^z) = zI_D(g)$, $I_D(g) = 0$ for $g \in [G, G]$.

(2) $I_D(g)$ is calculated from the fixed point
data by using the Atiyah-Singer's
theorem if g is periodic.

Using the properties above, we can use I_D
as an obstruction to the existense of G-actions.

Example Let p be an odd prime number
and r a natural number defined by

$$r = r_p(k) = kp + \frac{p-1}{2} \quad \left(k \in \mathbf{Z}\,,\ 0 \le k \le \frac{(p-1)(p-2)}{2p} \right).$$

Then it follows from a result of Glover-Mislin (1987) that

1. $\mathbf{Z}_p \subset \Gamma^r$
   $\left(\begin{array}{l}\text{namely, the compact Riemann surface } \Sigma^r \text{ of genus} \\ r = r_p(k) \text{ admits an action of the cyclic group } \mathbf{Z}_p.\end{array}\right)$
2. Fixed point set of $g \in \mathbf{Z}_p$ consists of 3 points.

Suppose that the fixed point set of g consists of q_1, q_2, q_3 and that $g \cdot v = \xi_p^{\tau_i}\, v$ for $v \in T_{q_i}\Sigma^r$.
Then, the value $I_{D_\ell}(g)$ for the $\otimes^\ell T\Sigma^r$-valued Dolbeault operator D_ℓ on Σ^r is calculated as follows:

$$12pI_{D_\ell}(g^z) \equiv F_{p,r}(z,\ell\,;\tau_1,\tau_2,\tau_3) \ \ \mathrm{mod}.12p$$

for $1 \le z \le p-1$ where

$$F_{p,r}(z,\ell\,;\tau_1,\tau_2,\tau_3)$$
$$= 6(p-1)(1-r)(2\ell+1)$$
$$+\sum_{i=1}^{3}\left\{\begin{array}{l} z\tau_i(p-1)(7p-11) \\ +6 \displaystyle\sum_{j=\left[\frac{(\ell+1)z\tau_i}{p}\right]+1}^{\left[\frac{(\ell+p+1)z\tau_i}{p}\right]} f_p\left(\left[\frac{jp-1}{z\tau_i}\right]-\ell-1\right) \end{array}\right\}$$

$$\left(\ f_p(x) = x^2-(p-2)x-(p-1)^2\ \right)$$

【12】

For a prime number p, we call a finite group G a C_p group if the order of the commutator subgroup $[G, G]$ is a multiple of p. Note that

1. C_p group is a non-abelian finite group
2. Any non-abelian finite group is a C_p group for some prime number p
3. $[C_p, C_p]$ contains an elment g of order p (Cauchy)

Assume that a C_p group G acts on Σ^r for $r = r_p(k)$. Then the property 3 above implies the existence of natural numbers (τ_1, τ_2, τ_3) $(1 \leq \tau_1,\ \tau_2,\ \tau_3 \leq p-1)$ which satisfy the following condition:

$$F_{p,r}(z,\ell\,;\tau_1,\tau_2,\tau_3)\equiv 0 \text{ mod.} 12p$$

$$\text{for } 1\le z\le p-1\,,\ 0\le \ell\le p-1\,.$$

$$(\Longleftrightarrow\ I_{D_\ell}(g^z)=zI_{D_\ell}(g)=0)$$

But direct computation using a computer shows that there does not exist such (τ_1,τ_2,τ_3) when

$$p=5,\ 11,\ 17,\ 23,\ 29,\ 41,\ 47,\ 53,\ 59,\ 71,\ 83,\ 89.$$

This result implies that any C_p group can not act on Σ^r for $r=r_p(k)$ if p is in the list above.

For example, C_p group can not act on Σ^r when

$$(p,r) = (5,2)\,,\ (5,7)\,,$$
$$(p,r) = (89, 44+89k)\ (0 \le k \le 43)\,.$$

Remark Let D_{2p} be the dihedral group of order $2p$.
Then Bujalance-Cirre-Gamboa-Gromadzki (2003) shows that $\min\{r \mid D_{2p} \subset \Gamma^r\} = p-1$.
For example, $D_{2\cdot 5} \subset \Gamma^4$, $\quad D_{2\cdot 89} \subset \Gamma^{88}$